SUMS OF SQUARES ON REDUCIBLE REAL CURVES

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Abstract. We ask whether every polynomial function that is non-negative on a real algebraic curve can be expressed as a sum of squares in the coordinate ring. Scheiderer has classified all irreducible curves for which this is the case. For reducible curves, we show how the answer depends on the configuration of the irreducible components and give complete necessary and sufficient conditions. We also prove partial results in the more general case of finitely generated preorderings and discuss applications to the moment problem for semialgebraic sets.

Introduction

Let V be an affine variety over $\mathbb{R}$, and let $V(\mathbb{R})$ be its set of real points. Can every polynomial function that is non-negative (*psd*) on $V(\mathbb{R})$ be expressed as a sum of squares (*sos*) in the coordinate ring $\mathbb{R}[V]$? If this is the case, we say that *psd=sos* in $\mathbb{R}[V]$. It has been known since Hilbert (1888) that not every real psd polynomial in n variables can be expressed as a sum of squares of real polynomials, unless $n = 1$. On the one hand, Hilbert's negative result can be generalised to show that psd$\neq$sos as soon as the dimension of $V(\mathbb{R})$ is at least 3; see Scheiderer [11]. This leaves the question of psd vs. sos for varieties of dimension at most 2 where the answer depends on the geometry of the curve or surface in question. On the other hand, weaker statements can be proved in any dimension: For example, a famous theorem of Schmüdgen implies that if $V(\mathbb{R})$ is compact, then every strictly positive element of $\mathbb{R}[V]$ (i.e. every psd function without real zeros) is a sum of squares; see Schmüdgen [16] or Prestel and Delzell [10], Thm. 5.1.17. This and similar statements of an approximative nature have applications to polynomial optimization and functional analysis. (General references on this topic are the books of Marshall [6], Prestel and Delzell [10], and Scheiderer's survey [15].)

The case when V is one-dimensional (a real algebraic curve) is completely understood if V is irreducible, i.e. if V cannot be expressed as a union of two non-empty curves. Scheiderer has shown that psd=sos in the coordinate ring $\mathbb{R}[C]$ of a non-singular irreducible real affine curve C if and only if C is rational or if C admits a non-constant bounded function (a function $f \in \mathbb{R}[C]$, $f \notin \mathbb{R}$, such that $|f| \leqslant n$ on $C(\mathbb{R})$ for some $n \in \mathbb{N}$). He calls the latter property *virtual compactness* since the problem of psd vs. sos for such curves always reduces to the compact situation. The singular case, too, is completely understood. For all these results, see Scheiderer [11] and [12]. It is the goal of this paper to extend them to the reducible case. Motivation, other than a desire for completeness, comes from Schmüdgen's fibration theorem, which can be combined with our results to study the one- and two-dimensional moment problem of functional analysis.

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For the purpose of this introduction, let us consider the case of plane curves, which is conceptually simpler. Let $F \in \mathbb{R}[x,y]$ be a square-free polynomial with real coefficients, and let

$$C(\mathbb{R}) = \left\{P \in \mathbb{R}^2 \mid F(P) = 0\right\}$$

be the set of real points of the (affine) plane curve C determined by F. We will use the informal notation $C = \{F = 0\}$. If $F = F_1 \cdots F_r$ is the factorization of F in $\mathbb{R}[x,y]$ into its irreducible factors, the curves $C_i = \{F_i = 0\}$ yield the decomposition $C = C_1 \cup \cdots \cup C_r$ of C into its ($\mathbb{R}$-)irreducible components. The coordinate ring $\mathbb{R}[C]$ is just the residue class ring $\mathbb{R}[x,y]/(F)$. Write $F = \sum F^{(j)}$ with $F^{(j)}$ homogeneous of degree j. If the point $(0,0)$ lies on C, recall that $(0,0)$ is called an *ordinary double point* of C if either $F^{(1)} \neq 0$ or $F^{(1)} = 0$ and $F^{(2)}$ is a product of two distinct linear factors (up to scalar multiples) in $\mathbb{R}[x,y]$. Geometrically, this means that $(0,0)$ is either a non-singular point of C or a singular point contained in exactly two branches with linearly independent tangents at $(0,0)$. With a change of coordinates, this notion extends to any real point of the plane.

If C is the union of two irreducible components C_1 and C_2 that intersect at a real ordinary double point P and have no further intersection points, it is easy to describe polynomial functions on C in terms of polynomial functions on C_1 and C_2, namely

$$\mathbb{R}[C] \cong \left\{(f,g) \in \mathbb{R}[C_1] \times \mathbb{R}[C_2] \;\middle|\; f(P) = g(P)\right\}.$$

Now if (f,g) is non-negative on $C(\mathbb{R})$ and $f = \sum^n f_i^2$ and $g = \sum^n g_i^2$ are sums of squares in $\mathbb{R}[C_1]$ resp. $\mathbb{R}[C_2]$, then $(f,g) = \sum^n (f_i, g_i)^2$ in $\mathbb{R}[C_1] \times \mathbb{R}[C_2]$, and we must only see to it that $f_i(P) = g_i(P)$ for all i. This can easily be done (Prop. 2.2). Thus, for example, psd=sos in $\mathbb{R}[x,y]/(xy)$, which is the coordinate ring of two intersecting lines. We show that either a curve can be build up inductively from irreducible (or compact reducible) curves in this simple way, or else there exists some obstacle that prevents psd=sos. The main result, which completes Scheiderer's classification of affine curves for which psd=sos holds, is the following:

Theorem. *Let C be an affine curve over $\mathbb{R}$, and let C' be the union of all irreducible components of C that do not admit any non-constant bounded polynomial function. Then psd=sos in $\mathbb{R}[C]$ if and only if the following conditions are satisfied:*

1. *All real singularities of C are ordinary multiple points with independent tangents.*
2. *All intersection points of C are real.*
3. *All irreducible components of C' are non-singular and rational.*
4. *The configuration of the irreducible components of C' contains no loops.*

Here, an ordinary multiple point with independent tangents is just the proper higher-dimensional analogue of an ordinary double point. The last condition will be made precise in section 3. Some examples:

1. Let $C = \{xy(1-x-y) = 0\}$, three lines forming a triangle. This constitutes the kind of loop that condition (4) of the theorem forbids. For a concrete example of a psd function that is not a sum of squares in $\mathbb{R}[C]$, write $\mathbb{R}[C] \cong \left\{(f,g,h) \in \mathbb{R}[t] \times \mathbb{R}[u] \times \mathbb{R}[v] \,\middle|\, f(0) = g(1), f(1) = h(0), g(0) = h(1)\right\}$, and put $f = 2t-1$, $g = 2u-1$, $h = 2v-1$. Then (f^2, g^2, h^2) is an element of $\mathbb{R}[C]$ which is clearly psd. But if $f^2 = \sum f_i^2$, with all $f_i \neq 0$, then $f_i\left(\frac{1}{2}\right) = 0$ and $\deg f_i = 1$, so $f_i = a_i f$ with constants $a_i \in \mathbb{R}$ for all i. The same is true for g and h. Therefore, if we had $(f^2, g^2, h^2) = \sum s_i^2$ in $\mathbb{R}[C]$, then each s_i

would have to be given as $s_i = (a_i f, b_i g, c_i h)$ with constants $a_i, b_i, c_i \neq 0$. Such a triple can never define a function on C, for if $a_i f(0)$ and $b_i g(1)$ have the same sign and $a_i f(1)$ and $c_i h(0)$ have the same sign, then $b_i g(0)$ and $c_i h(1)$ will have different signs.

(2) Consider the family of curves $C_a = \{(y - x^2)(y - a) = 0\}$ for $a \in \mathbb{R}$, the union of a parabola and a line. Again, psd$\neq$sos in $\mathbb{R}[C_a]$ for any value of the parameter a, but for varying reasons: If $a < 0$, then the parabola and the line intersect at a pair of distinct complex-conjugate points so that condition (2) of the theorem is not met. For $a = 0$, the intersection of the line and the parabola in the origin is not an ordinary double point, violating condition (1). And if $a > 0$, the line and the parabola intersect at two distinct real points which violates condition (4). (In all three cases, one can argue directly in a similar way as in the previous example.)

(3) For a positive example, let C be the curve $\{(x^2 + y^2 - 1)y = 0\}$, a line intersecting a circle in two distinct real points. Since the real points of the circle $\{x^2 + y^2 = 1\}$ are compact, condition (4) is empty, and psd=sos in $\mathbb{R}[C]$.

A brief overview of the structure of this paper: After a few preliminaries in section 1, we introduce general techniques for dealing with sums of squares on reducible varieties, that are not peculiar to curves, in section 2. But even if we assume that irreducible components intersect at only finitely many points, there is not much to be said here in complete generality. The most useful results are the basic Prop. 2.2, as well as Prop. 2.7 which deals with a union of two subvarieties one of which is assumed compact. Section 3 is the longest, mostly devoted to the proof of the above theorem (Thm. 3.15) in several steps. In section 4, we look at possible generalisations to the case of finitely generated preorderings in place of sums of squares. We give sufficient conditions for a reasonably large class of examples (Prop. 4.5), but many cases remain open. In section 5, we briefly explain how our results can be applied to the moment problem of functional analysis.

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1. Preliminaries

In this section, we fix notations and briefly discuss some notions from real algebra and geometry, as well as some general facts concerning reducible algebraic varieties and curves. From algebraic geometry, we need only basic concepts and results, but because of the real ground field and a few non-reduced phenomena, it is convenient to work with schemes.

Let always k be a field and R a real closed field, for example the field of real numbers denoted by $\mathbb{R}$.

1.1. A *variety* over k is a reduced separated scheme of finite type over $\mathrm{Spec}(k)$, not necessarily irreducible. A *curve* is a variety all of whose irreducible components have (Krull-)dimension 1. If V is a variety over k and K/k a field extension, we denote by $V(K)$ the set of K-valued points of V. We frequently consider $V(k)$ as a subset of V by identifying $V(k)$ with the set of points of V with residue field k. We use the notation $k[V]$ for the coordinate ring $\mathcal{O}_V(V)$ of an affine k-variety V.

1.2. Let V be a variety over R. Any closed point $P \in V$ has residue field $\kappa(P) = R$ or $\kappa(P) = R(\sqrt{-1})$. In the first case, the point P is called *real*, in the second *non-real*. The set $V(R)$ of real points is equipped with the semialgebraic topology, induced by the ordering of R, which is the euclidean topology if $R = \mathbb{R}$. Unless explicitly stated otherwise, topological statements about subsets of $V(R)$ will refer to that topology. If C is a non-singular curve over R, a *divisor* on C is any finite $\mathbb{Z}$-linear combination of closed points. The *degree* of a divisor $\sum_{P\in C} n_P P$ is defined as $\sum n_P[\kappa(P) : R]$; in other words, non-real points are counted with multiplicity 2. If C is irreducible and $P \in C$ is a closed point, we denote by ord_P the discrete valuation of the function field $R(C)$ corresponding to the order of vanishing at P. For a rational function $f \in R(C)$, we write $\mathrm{div}_C(f)$ for the divisor $\sum_{P\in C} \mathrm{ord}_P(f)P$ of zeros and poles of f.

Definition 1.3. Let V be an affine R-variety, and let V_1, V_2 be closed subvarieties of V. Assume that V_1 and V_2 intersect at only finitely many points $P_1, \ldots, P_r$ of V. We say the intersection of V_1 and V_2 is *transversal* or that V_1 and V_2 intersect transversally if $\mathcal{I}_{V_1} + \mathcal{I}_{V_2} = \bigcap_{i=1}^r \mathfrak{m}_{P_i}$.

Here, I_W denotes the vanishing ideal of a subvariety W in $R[V]$, and $\mathfrak{m}_P$ denotes the vanishing ideal of a point P.

Lemma 1.4. *If V is the union of two closed subvarieties V_1 and V_2 that intersect at finitely many points $P_1, \ldots, P_r$, then the intersection of V_1 and V_2 is transversal if and only if the diagonal homomorphism*

$$R[V] \to \big\{(f, g) \in R[V_1] \times R[V_2] \mid \forall i \in \{1, \ldots, r\} \colon f(P_i) = g(P_i)\big\}$$

given by $f \mapsto (f|_{V_1}, f|_{V_2})$ is an isomorphism.

Proof. Let φ be the map $R[V] \to R[V_1] \times R[V_2]$ given by $f \mapsto (f|_{V_1}, f|_{V_2})$. We have $\mathcal{I}_{V_1} \cap \mathcal{I}_{V_2} = (0)$, because V is the union of V_1 and V_2, hence φ is injective. The image of φ consists of all elements $(f|_{V_1}, g|_{V_2}) \in R[V_1] \times R[V_2]$, $f, g \in R[V]$, such that $f - g \in \mathcal{I}_{V_1} + \mathcal{I}_{V_2}$. On the other hand, $(f, g) \in R[V] \times R[V]$ satisfies $f(P_i) = g(P_i)$ for all $i \in \{1, \ldots, r\}$ if and only if $f - g \in \bigcap_{i=1}^r \mathfrak{m}_{P_i}$. Thus the image of φ has the desired form if and only if $\mathcal{I}_{V_1} + \mathcal{I}_{V_2} = \bigcap_{i=1}^r \mathfrak{m}_{P_i}$ which proves the claim. □

1.5. Let C be a curve over k. A closed point $P \in C$ with residue field K is called an *ordinary multiple point with independent tangents* if the completed local ring $\widehat{\mathcal{O}}_{C,P}$ is isomorphic to $K[[x_1, \ldots, x_n]]/(x_i x_j \mid 1 \leqslant i < j \leqslant n)$ for some n.
Every non-singular point on a curve is an ordinary multiple point with independent tangents ($n = 1$). A point on a plane curve is an ordinary multiple point with independent tangents if and only if it is an ordinary double point as defined in the introduction ($n = 2$).

Lemma 1.6. *For any field K and any $n \geqslant 1$, there is an isomorphism*

$$\begin{aligned} &K[[x_1, \ldots, x_n]]/(x_i x_j \mid 1 \leqslant i < j \leqslant n) \\ &\cong \big\{(f_1, \ldots, f_n) \in K[[t_1]] \times \cdots \times K[[t_n]] \bigm| f_1(0) = \cdots = f_n(0)\big\}. \end{aligned}$$

given by the the map $\varphi \colon K[[x_1, \ldots, x_n]]/(x_i x_j \mid i, j) \longrightarrow \prod_i K[[t_i]]$ that sends the class of $f \in K[[x_1, \ldots, x_n]]$ to $\big(f(t_1, 0, \ldots, 0), \ldots, f(0, \ldots, 0, t_n)\big)$,

Proof. One checks that φ is well-defined and injective. To see that the image is as claimed, let $(f_1, \ldots, f_n)$ be an element of the right-hand side such that $f_1(0) = \cdots = f_n(0)$. Then $\varphi\big(f_1(x_1) + \cdots + f_n(x_n) - (n-1)f_1(0)\big) = (f_1, \ldots, f_n)$. □

Corollary 1.7. *Let C be a curve with irreducible components $C_1, \ldots, C_m$ over a field k. Assume that all singularities of $C_1, \ldots, C_m$ are ordinary multiple points with independent tangents. The following are equivalent:*

(1) *All singularities of C are ordinary multiple points with independent tangents.*
(2) *The closed subcurves C_i and $C_i' = \bigcup_{j \neq i} C_j$ intersect transversally for every $i = 1, \ldots, m$.*
(3) *The diagonal homomorphism $k[C] \to \prod_{i=1}^m k[C_i]$ given by $f \mapsto (f|_{C_i})_{i=1,\ldots,m}$ induces an isomorphism between $k[C]$ and*

$$\Big\{(f_i) \in \prod_i k[C_i] \;\Big|\; f_i(P) = f_j(P) \text{ for all } P \in C_i \cap C_j,\ i, j = 1, \ldots, m\Big\}.$$

(where $C_i \cap C_j$ is the usual, set-theoretic intersection).

Proof. The equivalence of (2) and (3) is immediate from Lemma 1.4. Assume that (1) holds and write φ for the diagonal homomorphism $k[C] \to \prod_{i=1}^m k[C_i]$. For any closed point $P \in C$, consider the induced map $\varphi_P \colon \widehat{\mathcal{O}}_{C,P} \to \prod_i \widehat{\mathcal{O}}_{C_i,P}$ on completed local rings (note that $\widehat{\mathcal{O}}_{C_i,P} = 0$ if $P \notin C_i$). Since P is an ordinary multiple point with independent tangents, it follows from Lemma 1.6 that φ_P is an isomorphism. So φ is an isomorphism after completion with respect to any maximal ideal of $k[C]$, hence it is an isomorphism (see for example Bourbaki [2], §3, no. 5, Cor. 5).

Conversely, if (3) holds and $P \in C$ is any closed point, then P is an ordinary multiple point with independent tangents on each irreducible component of C passing through P by assumption, hence by Lemma 1.6, P must also be an ordinary multiple point with independent tangents on C. □

1.8. A field is called *real* if it possesses an ordering, i.e. a linear order of the set K that respects addition and multiplication. A valuation $v \colon K^\times \to \Gamma$ of a field K is called *real* if its residue field is real. The following simple lemma is of fundamental importance to the study of sums of squares on real varieties. It generalises the observation that "leading terms cannot cancel" in a sum of squares of real polynomials.

Lemma 1.9. *Let K be a field equipped with a real valuation v, and let $a_1, \ldots, a_r \in K$. Then $v(a_1^2 + \cdots + a_r^2) = 2\min\{v(a_1), \ldots, v(a_r)\}$.*

Proof. By the Baer-Krull theorem (see for example Bochnak, Coste, and Roy [1], Thm. 10.1.10), there exists an ordering of K that is compatible with the valuation v. Then $\sum_j a_j^2 \geqslant a_i^2$ in K implies $v(\sum_j a_j^2) \leqslant 2v(a_i)$ for all i. The opposite inequality holds trivially. □

1.10. An irreducible variety V over R is called *real* if the following equivalent conditions are satisfied: (1) $V(R)$ is Zariski-dense in V. (2) $V(R)$ contains a non-singular point of V. (3) The semialgebraic dimension of $V(R)$ coincides with the (Krull-) dimension of V. (4) The function field $R(V)$ is real (see [8], Prop. A.1 or Bochnak, Coste, and Roy [1], section 7.6). A reducible variety over R is called real if all its irreducible components are real.

1.11. Let V be an affine R-variety, and let S be a semialgebraic subset of $V(R)$. We write

$$B_V(S) = \big\{f \in R[V] \mid \exists \lambda \in R\ \forall x \in S \colon |f(x)| \leqslant \lambda\big\}$$

for the ring of bounded functions on S. Its size can be seen as a measure for the "compactness" of S. Mostly, we will be interested in the case $S = V(R)$ for which

we use the notation

$$B(V) = B_V(V(\mathbb{R})).$$

We collect what we will need about rings of bounded functions on curves in the following lemma:

Lemma 1.12. *Let C be an affine curve with irreducible components $C_1, \dots, C_m$ over R. Let S be a semialgebraic subset of $C(R)$ and write $S_i = S \cap C_i(R)$.*

(1) *There exists an open-dense embedding of C into a projective curve X such that the finitely many points of $X \setminus C$ are non-singular. The embedding $C \hookrightarrow X$ is unique up to isomorphism.*
(2) *Let T be the union of all points of $X \setminus C$ that are either non-real, or real but not contained in the closure of S in $X(R)$. Put $\widetilde{C} = X \setminus T$. Then $B_C(S) \cong \mathcal{O}_{\widetilde{C}}(\widetilde{C})$.*
(3) *If C is irreducible, then $B_C(S) \neq R$ if and only if $T \neq \emptyset$.*
(4) *If $B_{C_i}(S_i) \neq R$ for all $i = 1, \dots, m$, then $\widetilde{C} \cong \operatorname{Spec}(B_C(S))$.*
(5) *If $B_{C_j}(S_j) \neq R$ for some $j \in \{1, \dots, m\}$, there exists a non-constant element $f \in B_C(S)$ such that $f|_{C_i} = 0$ for all $i \neq j$. In particular, if C is connected, then $B_C(S) = R$ if and only if $B_{C_i}(S_i) = R$ for all $i = 1, \dots, m$.*
(6) *If $h \in B_C(S)$ vanishes at all points of $\widetilde{C} \setminus C$, then for every $f \in R[C]$ there exists $N \geqslant 0$ such that $h^N f \in B_C(S)$.*
(7) *If $B_C(S) = R$ and $f_1, \dots, f_r \in R[C]$ are such that $f_1^2 + \cdots + f_r^2 \in R$, then $f_1, \dots, f_r \in R$.*

Remark 1.13. We will mostly use this lemma in the global case $S = C(R)$. Note that, since the points of $X \setminus C$ in (1) are non-singular by definition, $C(R)$ is dense in $X(R)$. So if $S = C(R)$, then T in (2) consists exactly of the non-real points of $X \setminus C$; in particular, $\widetilde{C}(R) = X(R)$ is semialgebraically compact.

Proof. (1) For the existence of $C \hookrightarrow X$, start with any embedding of C into affine space, take the closure X_0 in the corresponding projective space, and apply resolution of singularities for curves to the finitely many points of $X_0 \setminus C$. If $C \hookrightarrow X_1$, $C \hookrightarrow X_2$ are two such embeddings, the identity map $C \to C$ induces a birational morphism $X_1 \dashrightarrow X_2$ which is an isomorphism since X_1 and X_2 are projective and all points of $X_1 \setminus C$ and $X_2 \setminus C$ are non-singular (see for example Hartshorne [3], Prop. 6.8).

(2) Since X is projective, $X(R)$ is semialgebraically compact. Hence so is the closure $\overline{S}$ of S in $X(R)$. Therefore, $\overline{S} \subset \widetilde{C}(R)$ implies $\mathcal{O}_{\widetilde{C}}(\widetilde{C}) \subset B_C(S)$. For the converse, if $f \in R[C]$, $f \notin \mathcal{O}_{\widetilde{C}}(\widetilde{C})$, then f has a pole at a point of $\overline{S} \setminus S$. It is easy to see that f cannot be bounded on S (see also [8], Lemma 1.8).

(3) $B_C(S) = R$ if and only if there does not exist a rational function $f \in R(C)$ with poles only at points of T. This only happens if $T = \emptyset$, by the Riemann-Roch theorem.

(4) In general, if V is any variety, then $V \cong \operatorname{Spec}(\mathcal{O}_V(V))$ if and only if V is affine. But an irreducible curve is either affine or projective, and the hypothesis $B_{C_i}(S_i) \neq R$ for all $i = 1, \dots, m$ implies that none of the $\widetilde{C}_i$ is projective, by (3) (where $\widetilde{C}_i$ is defined for C_i as $\widetilde{C}$ for C in (2)).

(5) Let $j \in \{1, \dots, m\}$ with $B_{C_j}(S_j) \neq R$. Write $C' = \bigcup_{i \neq j} C_i$. To find f as in the claim, let J be the vanishing ideal of C_j in $R[C]$, so that $R[C_j] = R[C]/J$, and let I be the vanishing ideal of C' in $R[C]$. The ideal $I_j = (I + J)/J$ of $R[C_j]$ is non-zero, since $C_j \nsubseteq C'$. Therefore, the residue class ring $R[C_j]/I_j$ is zero-dimensional, thus it is a finite-dimensional R-vector space. On the other hand,

$B_{C_j}(S_j)$ is isomorphic to the coordinate ring of an affine curve by (2) and (4) and is therefore an infinite-dimensional subspace of $R[C_j]$. It follows that $I_j \cap B_{C_j}(S_j)$ is also infinite-dimensional. Hence there exists $f \in I$ such that $f|_{C_j} \in B_{C_j}(S_j) \setminus R$, and any such f will do what we want.

If C is connected, a non-constant function on C that is bounded on S must be non-constant on some C_i and bounded on S_i, so $B_{C_i}(S_i) = R$ for all $i = 1, \dots, m$ implies $B_C(S) = R$.

(6) We have $B_C(S) = \bigcap B_C(S_i)$. Therefore, we may assume that C is irreducible. Write $\widetilde{C} \setminus C = \{P_1, \dots, P_r\}$. Statement (2) says that $0 \neq f \in R[C]$ lies in $B(C)$ if and only if $\mathrm{ord}_{P_i}(f) \geqslant 0$ for all $i = 1, \dots, r$. We may assume $h \neq 0$. Since $\mathrm{ord}_{P_i}(h) > 0$ for all $i = 1, \dots, r$ by hypothesis, it follows that there exists $N \geqslant 0$ such that $\mathrm{ord}_{P_i}(h^N f) = N \cdot \mathrm{ord}_{P_i}(h) + \mathrm{ord}_{P_i}(f) \geqslant 0$ holds for all $i = 1, \dots, r$.

(7) By (5), $B_C(S) = R$ implies $B_{C_i}(S_i) = R$ for all irreducible components C_i of C. Thus we may assume that C is irreducible. Let $P \in X \setminus C$. Then P must be real by (3), hence the corresponding valuation ord_P is a real valuation of the function field $R(C)$. Thus if $f_1^2 + \dots + f_r^2 \in R$, then $\mathrm{ord}_P(f_i) = 0$ for all $i = 1, \dots, r$ by Lemma 1.9. It follows that the f_i have no poles on the complete curve X, i.e. they are contained in the intersection of all valuation rings of $R(C)$, which is R. □

Definition 1.14. The points of $X \setminus C$ with $C \hookrightarrow X$ as in the lemma, are called the *points at infinity* of C. The set of real points $C(R)$ (or, loosely speaking, the curve C itself) is called *virtually compact* if $B(C_i) \neq R$ holds for every irreducible component C_i of C or, equivalently, if every irreducible component of C has a non-real point at infinity (see Def. 4.8. in Scheiderer [12]).

1.15. We briefly fix notations and terminology for preorderings and semialgebraic sets that will be used in section 4: Let A be a ring. A *preordering* of A is a subset T of A that is closed under addition and multiplication and contains all squares of elements of A. Given a subset $\mathcal{H}$ of A, the *preordering generated by* $\mathcal{H}$ is the intersection of all preorderings of A containing $\mathcal{H}$ and is denoted by $\mathrm{PO}_A(\mathcal{H})$ or just $\mathrm{PO}(\mathcal{H})$. If $\mathcal{H} = \{h_1, \dots, h_r\}$ is finite, the generated preordering has a simple explicit description:

$$\mathrm{PO}_A(\mathcal{H}) = \left\{ \sum_{i \in \{0,1\}^r} s_i \underline{h}^i \mid s_i \in \sum A^2 \right\},$$

where we use the notation $\underline{h}^i = h_1^{i_1} \cdots h_r^{i_r}$. A preordering T of A is called finitely generated if there exists a finite subset $\mathcal{H}$ of A such that $T = \mathrm{PO}(\mathcal{H})$.

If V is an affine R-variety with coordinate ring $R[V]$ and S a subset of $V(R)$, the psd-cone of S

$$\mathcal{P}_V(S) = \big\{ f \in R[V] \mid \forall x \in S \colon f(x) \geqslant 0 \big\}$$

is a preordering of $R[V]$. Conversely, given a subset $\mathcal{H}$ of $R[V]$, we write

$$\mathcal{S}(\mathcal{H}) = \big\{ x \in V(R) \mid \forall h \in \mathcal{H} \colon h(x) \geqslant 0 \big\}.$$

A subset S of $V(R)$ is called *basic closed* if there exists some finite subset $\mathcal{H}$ of $R[V]$ such that $S = \mathcal{S}(\mathcal{H})$. A finitely generated preordering T of $R[V]$ is called *saturated* if $T = \mathrm{PO}(\mathcal{S}(T))$.

If T is a preordering of $R[V]$ and Z a closed subvariety of V with vanishing ideal $\mathcal{I}_Z \subset R[V]$, we denote the induced preordering $(T + \mathcal{I}_Z)/\mathcal{I}_Z$ of $R[Z] = R[V]/\mathcal{I}_Z$ by $T|_Z$.

2. Sums of squares on reducible varieties: Generalities

Let always R be a real closed field, and let V be an affine R-variety with coordinate ring $R[V]$. We say that *psd=sos in* $R[V]$ if for every $f \in R[V]$ with $f(P) \geqslant 0$ for all $P \in V(R)$ there exist $f_1, \ldots, f_n \in R[V]$ such that $f = f_1^2 + \cdots + f_n^2$. This is what is known in the irreducible case:

2.1. (1) If V is zero-dimensional, then $R[V]$ is a direct product of copies of R and $R(\sqrt{-1})$, so that psd=sos in $R[V]$.

(2) If $R = \mathbb{R}$ and V is an irreducible affine curve, Scheiderer has given necessary and sufficient conditions for psd=sos in $\mathbb{R}[V]$; see Cor. 3.6 and Thm. 3.9 below.

(3) If $R = \mathbb{R}$ and V is a non-singular irreducible affine surface over $\mathbb{R}$ such that $V(\mathbb{R})$ is compact, then psd=sos in $\mathbb{R}[V]$ (Scheiderer [14], Cor. 3.4).

(4) There exist examples of non-singular irreducible affine surfaces V over $\mathbb{R}$ such that $V(\mathbb{R})$ is not compact and psd=sos in $\mathbb{R}[V]$ (Scheiderer [14], Remark 3.15, and [8], section 4.4).

(5) If V is irreducible and not real, then it is not hard to show that psd=sos in $R[V]$ if and only if $V(R) = \emptyset$; see also Cor. 2.10 below.

We now turn to reducible varieties. To avoid confusion, note that we will be looking at three kinds of components of V: *Irreducible* components of V (for the Zariski-topology); *connected* components of V (again for the Zariski-topology); and occasionally connected components of $V(R)$ (for the semialgebraic topology which is the euclidean topology for $R = \mathbb{R}$). Any irreducible component is connected, and any connected component is a union of irreducible components. But even if V is irreducible, $V(R)$ need not be connected.

Question. Let $V_1, \ldots, V_m$ be the irreducible components of V, and assume that psd=sos in $R[V_1], \ldots, R[V_m]$. Under what circumstances can we conclude that psd=sos in $R[V]$?

Remark. It is not *a priori* clear that psd=sos in $R[V]$ implies psd=sos in each $R[V_i]$, since a psd function on $V_i(R)$ need not extend to a psd function on $V(R)$. We will later see that this implication does indeed hold for curves (Prop. 3.3). It would seem that it should be true in general, but I do not know of a way to prove this.

Note that if V has connected components $W_1, \ldots, W_l$, then $R[V] \cong R[W_1] \times \cdots \times R[W_l]$ and psd=sos in $R[V]$ if and only if psd=sos in all $R[W_1], \ldots, R[W_l]$. Thus we can always assume that V is connected, and the interesting data is how the irreducible components $V_1, \ldots, V_m$ intersect. We begin with the simplest conceivable case:

Proposition 2.2. *Let V be an affine R-variety that is the union of two closed subvarieties V_1 and V_2 intersecting transversally in a single real point, and let $f \in R[V]$. If $f|_{V_1}$ and $f|_{V_2}$ are sums of squares in $R[V_1]$ resp. $R[V_2]$, then f is a sum of squares in $R[V]$.*

Proof. Write $V_1 \cap V_2 = \{P\} \in V(R)$. Using Lemma 1.4, we can identify $R[V]$ with the subring of $R[V_1] \times R[V_2]$ consisting of all pairs (f, g) such that $f(P) = g(P)$. Let $(f, g) \in R[V]$ be such that $f = \sum_{i=1}^n f_i^2$, $g = \sum_{i=1}^n g_i^2$ with $f_1, \ldots, f_n \in R[V_1]$ and $g_1, \ldots, g_n \in R[V_2]$ (not necessarily $\neq 0$). The vectors $v = \big(f_1(P), \ldots, f_n(P)\big)^t$ and $w = \big(g_1(Q), \ldots, g_n(Q)\big)^t$ in R^n have the same euclidean length since $f(P) = g(Q)$.

Therefore, there exists an orthogonal matrix $B \in O_n(R)$ such that $Bv = w$. Put $(\widetilde{f}_1, \dots, \widetilde{f}_n)^t = B \cdot (f_1, \dots, f_n)^t$, then $f = \sum \widetilde{f}_i^2$ and $(\widetilde{f}_i, g_i) \in R[V]$ for all i, hence $(f, g) = \sum_{i=1}^n (\widetilde{f}_i, g_i)^2$. □

Examples 2.3. (1) Let C be the plane curve $\{xy = 0\}$. Then psd=sos in $R[C]$ by the proposition.

(2) More generally, let V_1 and V_2 be any two affine R-varieties with $V_i(R) \neq \emptyset$. Fix points $P \in V_1(R)$ and $Q \in V_2(R)$, take the ring $A = \{(f, g) \in R[V_1] \times R[V_2] \mid f(P) = g(Q)\}$, and let V be the affine variety $\mathrm{Spec}(A)$. (To see that A is a finitely generated R-algebra, choose generators $x_1, \dots, x_n$ of $R[V_1]$ and $y_1, \dots, y_m$ of $R[V_2]$ such that $x_i(P) = 0$ and $y_i(Q) = 0$ for all i. Then A is generated by the elements $(x_i, 0), (0, y_i)$ and $(1, 1)$.) The variety V is V_1 and V_2 glued transversally along P and Q. If psd=sos in $R[V_1]$ and $R[V_2]$, it also holds in $R[V]$. A simple example would be given by two 2-dimensional spheres over $\mathbb{R}$ intersecting transversally.

If V_1 and V_2 intersect at more than one point, the statement of 2.2 becomes false in general, as we shall see later. However, we can still say something if $R = \mathbb{R}$ and $V_1(\mathbb{R})$ is compact. For the case of curves, we will also allow the slightly weaker condition of virtual compactness: Recall from 1.11 that $B(V)$ denotes the ring of bounded functions on an affine variety V; if C is a curve over $\mathbb{R}$, then $C(\mathbb{R})$ is called virtually compact if every irreducible component of C admits a non-constant bounded function; see 1.14.

We will need the existence of a certain kind of polynomial partition of unity adapted to points in the sense of the following Lemma. It is an easy consequence of a basic topological lemma due to Kuhlmann, Marshall, and Schwartz.

Lemma 2.4. *Let V be an affine $\mathbb{R}$-variety such that $V(\mathbb{R})$ is compact or, if V is a curve, virtually compact. Given finitely many distinct points $P_1, \dots, P_r \in V(\mathbb{R})$, there exist $h_1, \dots, h_r \in B(V)$ with the following properties:*

(1) $\sum_{i=1}^r h_i = 1$;
(2) $h_i \geqslant 0$ *on* $V(\mathbb{R})$;
(3) $h_i(P_j) = 0$ *for all* $i \neq j$ *(hence* $h_i(P_i) = 1$*).*

Proof. Assume $r \geqslant 2$, the case $r = 1$ being trivial. Let $W = \mathrm{Spec}(B(V))$. If $V(\mathbb{R})$ is compact, then $B(V) = \mathbb{R}[V]$ and $W = V$. If V is a curve and $V(\mathbb{R})$ is virtually compact, then the canonical morphism $V \to W$ induced by the inclusion $B(V) \subset \mathbb{R}[V]$ is an embedding of affine curves and $W(\mathbb{R})$ is compact (Lemma 1.12). It then suffices to prove the Lemma for W. It is therefore not restrictive to assume that $V(\mathbb{R})$ is compact.

Choose elements $g_1, \dots, g_r \in \mathbb{R}[V]$ such that $g_i(P_j) = 0$ for all $i \neq j$ and $g_i(P_i) \neq 0$. Since $(g_1^2 + \cdots + g_{r-1}^2)(P_i) = 0$ if and only if $i = r$, the Nullstellensatz gives an identity $a(g_1^2 + \cdots + g_{r-1}^2) + g = 1$ with $a, g \in \mathbb{R}[V]$ and $g(P_i) = 0$ for $i = 1, \dots, r-1$. It follows that $g_1^2 + \cdots + g_{r-1}^2$ and g have no common (complex) zeros on V. Hence again by the Nullstellensatz, there is an identity $s(g_1^2 + \cdots + g_{r-1}^2) + tg^2 = 1$, $s, t \in \mathbb{R}[V]$. Now since $V(\mathbb{R})$ is compact, Lemma 2.1 in [5] states that we can find such s, t with the additional property that s and t are strictly positive on $V(\mathbb{R})$. Thus $h_i = sg_i^2$ for $i \in \{1, \dots, r-1\}$ and $h_r = tg^2$ will do what we want. □

Corollary 2.5. *Let V be an affine $\mathbb{R}$-variety such that $V(\mathbb{R})$ is compact or, if V is a curve, virtually compact. Given finitely many distinct points $P_1, \dots, P_r \in V(\mathbb{R})$ and real numbers $a_1, \dots, a_r \in [-1, 1]$, there exists a function $g \in \mathbb{R}[V]$ such that*

$g(P_i) = a_i$ for all $i \in \{1, \dots, r\}$ and $|g(x)| \leqslant 1$ for all $x \in V(\mathbb{R})$. If $a_1, \dots, a_r$ are all non-negative, one can find such a g that is non-negative on $V(\mathbb{R})$, as well.

Proof. Take $h_1, \dots, h_r$ as in the lemma and put $g = \sum_{i=1}^r a_i h_i$. Then $g(P_i) = a_i$ and $|g(x)| \leqslant \sum_i |a_i| h_i(x) \leqslant \sum_i h_i(x) = 1$ for all $x \in V(\mathbb{R})$. If $a_1, \dots, a_r$ are non-negative, take $h \in \mathbb{R}[V]$ such that $h(P_i) = \sqrt{a_i}$ for all $i = 1, \dots, r$ and $|h| \leqslant 1$ on $V(\mathbb{R})$, and put $g = h^2$. □

Lemma 2.6. *Let V be an affine $\mathbb{R}$-variety such that $V(\mathbb{R})$ is compact or, if V is a curve, virtually compact. Assume that psd=sos in $\mathbb{R}[V]$. Let $P_1, \dots, P_r \in V(\mathbb{R})$ be distinct points, and let $f \in \mathbb{R}[V]$ be psd. Given $m \geqslant 1$ and vectors $a^{(1)}, \dots, a^{(r)} \in \mathbb{R}^m$ such that $||a^{(i)}|| = f(P_i)$ for all $i \in \{1, \dots, r\}$, there exist $n \geqslant m$ and $f_1, \dots, f_n \in \mathbb{R}[V]$ with the following properties:*

(1) *$f = \sum_{i=1}^n f_i^2$;*
(2) *$f_j(P_i) = a_j^{(i)}$ for all $i = 1, \dots, r$, $1 \leqslant j \leqslant m$;*
(3) *$f_j(P_i) = 0$ for all $i = 1, \dots, r$, $m+1 \leqslant j \leqslant n$.*

Proof. Note first that property (3) is automatic if (1) and (2) are satisfied. We first show that there exists $\widetilde{f} \in B(V)$ such that $0 \leqslant \widetilde{f} \leqslant f$ on $V(\mathbb{R})$ and such that $\widetilde{f}(P_i) = f(P_i)$ for all $i = 1, \dots, r$. If $V(\mathbb{R})$ is compact, we can take $\widetilde{f} = f$. If V is a curve and $V(\mathbb{R})$ is virtually compact, put $W = \operatorname{Spec}(B(V))$ and consider V as a subcurve of W via the embedding $V \hookrightarrow W$ induced by the inclusion $B(V) \subset \mathbb{R}[V]$ (Lemma 1.12). Let $Q_1, \dots, Q_s \in W(\mathbb{R})$ be the finitely many points of $W \setminus V$. By Cor. 2.5, there exists $h \in B(V)$ such that $h(Q_i) = 0$ for all $i = 1, \dots, s$, $h(P_i) = 1$ for all $i = 1, \dots, r$ and $0 \leqslant h \leqslant 1$ on $W(\mathbb{R})$. Then $h^N f \in B(V)$ for sufficiently large N (by Lemma 1.12 (6)), and we can take $\widetilde{f} = h^N f$ for such an N.

We have $f - \widetilde{f} \geqslant 0$ on $V(\mathbb{R})$ and $(f - \widetilde{f})(P_i) = 0$ for all $i = 1, \dots, r$. Since psd=sos in $\mathbb{R}[V]$, we can write $f = h_1^2 + \cdots + h_k^2 + \widetilde{f}$ with $h_1, \dots, h_k \in \mathbb{R}[V]$ and $h_j(P_i) = 0$ for all $i = 1, \dots, r$ and $j = 1, \dots, k$. It therefore suffices to prove the claim for $\widetilde{f}$.

Hence we may assume that f is bounded on $V(\mathbb{R})$. We may further assume that $f \neq 0$ and that $f \leqslant 1$, upon replacing f by $f \cdot \sup_{x \in V(\mathbb{R})}\{f(x)\}^{-1}$. We define $f_1, \dots, f_m$ recursively as follows: Apply Cor. 2.5 and choose $g_1 \in \mathbb{R}[V]$ with $|g_1| \leqslant 1$ on $V(\mathbb{R})$ and such that $g_1(P_i) = 0$ if $f(P_i) = 0$ and $g_1(P_i) = a_1^{(i)} \cdot f(P_i)^{-1}$ otherwise (for all $i = 1, \dots, r$). Put $f_1 = g_1 f$, then $f_1(P_i) = a_1^{(i)}$ (note that $f(P_i) = 0$ implies $a_1^{(i)} = 0$) and $f_1^2 \leqslant f$. Now since $f - f_1^2 \geqslant 0$, we can apply Cor. 2.5 again and choose $f_2 \in \mathbb{R}[V]$ such that $f_2(P_i) = a_2^{(i)}$ and $f_2^2 \leqslant f - f_1^2$. Continuing recursively in this manner will produce $f_1, \dots, f_m$ satisfying property (2) and such that $f - f_1^2 - \cdots - f_m^2 \geqslant 0$ holds on $V(\mathbb{R})$. Since psd=sos in $\mathbb{R}[V]$, there exist $n \geqslant m$ and elements $f_{m+1}, \dots, f_n \in \mathbb{R}[V]$ such that $f = \sum_{i=1}^n f_i^2$. □

Proposition 2.7. *Let V be an affine $\mathbb{R}$-variety, and assume that V is the union of two closed subvarieties V_1 and V_2 with the following properties:*

(1) *psd=sos in $\mathbb{R}[V_1]$;*
(2) *$V_1(\mathbb{R})$ is compact, or V_1 is a curve with $V_1(\mathbb{R})$ virtually compact;*
(3) *The intersection of V_1 and V_2 is finite and transversal;*
(4) *All points of $V_1 \cap V_2$ are real.*

Then every psd function $f \in \mathbb{R}[V]$ such that $f|_{V_2}$ is a sum of squares in $\mathbb{R}[V_2]$ is a sum of squares in $\mathbb{R}[V]$.

Proof. Write $V_1 \cap V_2 = \{P_1, \dots, P_r\} \subset V(\mathbb{R})$, $r \geqslant 0$. Using Lemma 1.4, we can identify $\mathbb{R}[V]$ with the subring of $\mathbb{R}[V_1] \times \mathbb{R}[V_2]$ consisting of all pairs (f, g) such

that $f(P_i) = g(P_i)$ for all $i = 1, \dots, r$. Let $F \in \mathbb{R}[V]$ be non-negative on $V(\mathbb{R})$ and such that $g = F|_{V_2} = g_1^2 + \dots + g_m^2$ for some $m \geqslant 1$, $g_1, \dots, g_m \in \mathbb{R}[V_2]$.

By the preceding lemma (applied with $a_j^{(i)} = g_j(P_i)$), there exist $n \geqslant m$ and elements $f_1, \dots, f_n \in \mathbb{R}[V_1]$ such that $f = F|_{V_1} = \sum_{j=1}^n f_j^2$, $f_j(P_i) = g_j(P_i)$ for all $j \leqslant m$, and $f_j(P_i) = 0$ for all $j > m$ ($i = 1, \dots, r$). It follows that $F_j = (f_j, g_j)$ for $j = 1, \dots, m$ and $F_j = (f_j, 0)$ for $j = m+1, \dots, n$ are elements of $\mathbb{R}[V]$, and that $F = (f, g) = \sum_{j=1}^n F_j^2$. □

We will mostly apply this proposition in the proof of our main result on curves (Thm. 3.15). As mentioned above, the condition that psd=sos in $\mathbb{R}[V_1]$ can only be satisfied if V_1 has dimension at most 2. But even in the two-dimensional case, hypothesis (3) is really too restrictive for the proposition to be of much use. One can produce examples though:

Example 2.8. Let $V_1 = \{x^2 + y^2 + z^2 = 1\}$, $V_2 = \{x = y = 0\}$, $V = V_1 \cup V_2$, a sphere and a line intersecting transversally in affine three-space with coordinates x, y, z over $\mathbb{R}$. Then psd=sos holds in $\mathbb{R}[V_1]$ by Scheiderer's results and in $\mathbb{R}[V_2]$ (elementary), thus it also holds in $\mathbb{R}[V]$.

We conclude this section with an observation in the non-real case:

Lemma 2.9 (Scheiderer [11], Lemma 6.3)**.** *Let A be a connected noetherian ring with* $\operatorname{Sper} A \neq \emptyset$*, and suppose that A is not real reduced. Then there exists $f \in A$ that vanishes identically on* $\operatorname{Sper} A$ *but is not a sum of squares in A.*

Corollary 2.10. *Let V be an affine R-variety and let V_r be the union of all real irreducible components, V_{nr} the union of all non-real irreducible components of V. Then psd=sos in $R[V]$ if and only if $V_{nr}(R) = \emptyset$, $V_r \cap V_{nr} = \emptyset$, and psd=sos in $R[V_r]$.*

Proof. Both $V_{nr}(R) \neq \emptyset$ or $V_r \cap V_{nr} \neq \emptyset$ imply that there exists a connected component V' of V such that V' is not real but $V'(R) \neq \emptyset$, so that psd≠sos in $R[V']$ by the lemma. As noted above, this implies psd≠sos in $R[V]$. Conversely, if $V_{nr}(R) = \emptyset$ and $V_r \cap V_{nr} = \emptyset$, then every element of $R[V_{nr}]$ is a sum of squares in $R[V_{nr}]$ by the real Nullstellensatz and $R[V] \cong R[V_r] \times R[V_{nr}]$, so that psd=sos in $R[V_r]$ is necessary and sufficient for psd=sos in $R[V]$. □

3. Sums of squares on curves

Let C be an affine curve over R with irreducible components $C_1, \dots, C_m$. We begin by showing that psd=sos in $R[C]$ implies psd=sos in $R[C_1], \dots, R[C_m]$. We will need the following

Proposition 3.1 (Scheiderer [12], Cor. 4.22)**.** *Let C be an affine curve over R. If C has a real singular point that is not an ordinary multiple point with independent tangents, then psd≠sos in $R[C]$.*

Lemma 3.2. *Let C be an affine curve over R all of whose real intersection points are ordinary multiple points with independent tangents. Let $C' \subset C$ be a closed subcurve of C. Then every psd function on C' can be extended to a psd function on C, i.e. every psd element of $R[C']$ is the image of a psd element of $R[C]$ under the restriction map $R[C] \to R[C']$.*

Proof. Let f be a psd function on C', and let D be the union of all irreducible components of C not contained in C'. Let $Z = C' \cap D$ be the scheme-theoretic intersection. We have to find a psd function on D that agrees with f on the closed

subscheme Z. If Z is supported on r real points and s non-real points, then the assumption on intersection points implies that the ring of regular functions $R[Z]$ of Z is a direct product

$$R[Z] \cong \underbrace{R \times \cdots \times R}_{r} \times A_1 \times \cdots \times A_s$$

with $R(\sqrt{-1}) \subset A_i$ for all $i \in \{1, \ldots, s\}$. This implies $\sum A_i^2 = A_i$ for all i, and since f takes only non-negative values at the real points of Z, it follows that the class of f in $R[Z]$ is a sum of squares in $R[Z]$. Therefore, it can be lifted to a sum of squares in $R[D]$ which, in particular, is psd on D. □

Example. The condition on intersection points cannot be dropped in general: If $C = \{y(y^2 - x^3) = 0\}$, the function x is psd on $C' = \{y^2 - x^3 = 0\}$ but cannot be extended to a psd function on C.

Proposition 3.3. *Let C be an affine curve over R. If psd=sos in $R[C]$, then psd=sos in $R[C']$ for every closed subcurve C' of C.*

Proof. If all real singularities of C are ordinary multiple points with independent tangents, then every psd function f on C' can be extended to a psd function g on C by Lemma 3.2. Then g is a sum of squares in $R[C]$ by hypothesis, hence f is a sum of squares in $R[C']$. If C has a real singular point that is not an ordinary multiple point with independent tangents, then psd≠sos in $R[C]$ by Prop. 3.1, so the statement is empty. □

Proposition 3.4. *Let C be a real curve over R, and assume that C has a non-real intersection point. Then psd≠sos in $R[C]$.*

Proof. Let C_1 and C_2 be two distinct irreducible components of C that intersect at a non-real point. By Prop. 3.3, it suffices to show that psd≠sos in $R[C_1 \cup C_2]$. We may therefore assume that $C = C_1 \cup C_2$.

Let $I_j = \mathcal{I}_{R[C]}(C_j)$ be the vanishing ideal of C_j inside $R[C]$ ($j = 1, 2$), and let Z be the closed subscheme of C_1 determined by the ideal $J = (I_1 + I_2)/I_1$ of $R[C_1] = R[C]/I_1$ (i.e. Z is the scheme-theoretic intersection of C_1 and C_2 inside C_1). Write $Z = S \cup T$ with closed subschemes S and T of C such that S is supported on real points and T is supported on non-real points. Let $I_S = \mathcal{I}_{R[C_1]}(S)$ and $I_T = \mathcal{I}_{R[C_1]}(T)$ be the vanishing ideals of S and T in $R[C_1]$, so that $J = I_S \cap I_T$. Since T is non-empty by hypothesis, we have $(0) \subsetneq I_T \subsetneq R[C_1]$. Put $A = R[C_1]/I_T^2$, and choose $h \in I_T \setminus I_T^2$. (Note that $I_T \neq I_T^2$ by the Krull intersection theorem; see Bourbaki [2], §3.2.) Since T has non-real support, we have $\sqrt{-1} \in A$, so every element of A is a sum of squares; in particular, $h + I_T^2$, the class of h in A, is a sum of squares in A. Therefore, the element $(0 + I_S, h + I_T^2) \in (R[C_1]/I_S) \times A$ is a sum of squares in $(R[C_1]/I_S) \times A \cong R[C_1]/(I_S \cap I_T^2)$. Thus there exists a sum of squares $f \in \sum R[C_1]^2$ that restricts to $(0 + I_S, h + I_T^2)$ modulo $I_S \cap I_T^2$. Since $h \in I_T$, we have $f \in J$.

Now choose $F \in I_2 \subset R[C]$ such that $F|_{C_1} = f$. Clearly, F is psd on $C(R)$. But we claim that F cannot be a sum of squares in $R[C]$. For if it were, say $F = \sum_{i=1}^r F_i^2$, $F_i \in R[C]$, it would follow that $F_i \in I_2$ for all $i = 1, \ldots, r$, since C_2 is real. This would imply $f = \sum_{i=1}^r f_i^2$ with $f_i = F_i|_{C_1}$. From $F_i \in I_2$ we could then conclude $f_i \in J$ for all $i = 1, \ldots, r$, hence $f \in J^2 \subset (I_S \cap I_T^2)$, contradicting the choice of h. □

In classifying all curves for which psd=sos, we first consider the virtually compact case, i.e. the case when every irreducible component admits a non-constant

bounded function. By Lemma 1.12, this is equivalent to saying that all irreducible components of C have a non-real point at infinity. For $R = \mathbb{R}$, Scheiderer has proved the following:

Theorem 3.5 (Scheiderer [12], Cor. 4.15)**.** *Let C be an affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ virtually compact. If C has no other real singularities than ordinary multiple points with independent tangents, then every psd function in $\mathbb{R}[C]$ with only finitely many zeros is a sum of squares in $\mathbb{R}[C]$.*

Corollary 3.6. *Let C be an irreducible affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ virtually compact. If C has no other real singularities than ordinary multiple points with independent tangents, then psd=sos in $\mathbb{R}[C]$.* □

For reducible curves, we have the following

Theorem 3.7. *Let C be a real affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ virtually compact. Then psd=sos in $\mathbb{R}[C]$ if and only if the following conditions are satisfied:*

(1) *All real points of C are ordinary multiple points with independent tangents.*
(2) *All intersection points of C are real.*

Proof. Sufficiency of (1) and (2) follows directly from Scheiderer's theorem: Let $C_1, \ldots, C_m$ be the irreducible components of C. By Cor. 1.7, we have

$$\mathbb{R}[C] \cong \left\{ (f_i) \in \prod_{i=1}^{m} \mathbb{R}[C_i] \;\middle|\; f_i(P) = f_j(P) \text{ for all } P \in C_i \cap C_j, 1 \leqslant i, j \leqslant m \right\}.$$

Let $f = (f_1, \ldots, f_m) \in \mathbb{R}[C]$ be psd. Upon relabelling, we may assume that f has only finitely many zeros on $C_1, \ldots, C_l$ and vanishes identically on $C_{l+1}, \ldots, C_m$ for some $0 \leqslant l \leqslant m$. Put $C' = \bigcup_{i=1}^{l} C_i$. Then $g = f|_{C'}$ is a sum of squares in $\mathbb{R}[C']$ by the preceding theorem, say $g = \sum g_i^2$, $g_i \in \mathbb{R}[C']$. For every $P \in C' \cap C_j$, $j > l$, we have $g(P) = f(P) = 0$, and since P is real by hypothesis, it follows that $g_i(P) = 0$ for all i. This implies that $f_i = \big(g_i|_{C_1}, \ldots, g_i|_{C_l}, 0, \ldots, 0\big)$ is a function on C, by the above description of $\mathbb{R}[C]$, and that $f = \sum f_i^2$.

Conversely, condition (1) is necessary by Prop. 3.1, condition (2) by Prop. 3.4. □

Example 3.8. Let $C = \{(x^2 + y^2 - 1)((x-1)^2 + y^2 - 1) = 0\}$, two intersecting circles in the plane. Then psd=sos in $R[C]$. But psd≠sos in $R[C]$ for $C = \{(x^2 + y^2 - 1)((x-3)^2 + y^2 - 1) = 0\}$, since the two circles intersect at a non-real point.

We now turn to the case furthest from virtual compactness, namely that of a real affine curve C that does not admit any bounded functions ($B(C) = R$). Again, the complete answer is known for the irreducible case, even for an arbitrary real closed ground field:

Theorem 3.9 (Scheiderer)**.** *Let C be an irreducible affine curve over R. Assume that $B(C) = R$. Then psd=sos in $R[C]$ if and only if C is isomorphic to an open subcurve of $\mathbb{A}^1_R$.*

Proof. It has already been noted in 1.11 that $B(C) = R$ holds if and only if all points of C at infinity are real. This is the way the hypothesis is stated in Scheiderer [12], Thm. 4.17. The proof for psd=sos for open subcurves of $\mathbb{A}^1_R$ easily reduces to the case of the polynomial ring in one variable; see also Scheiderer [11], Prop. 2.17. □

Note that an open subcurve of $\mathbb{A}^1_R$ is just the complement of finitely many points. Under the hypothesis $B(C) = R$, all those points must be real. These are exactly the non-singular, irreducible, rational, affine curves over R with $B(C) = R$.

For reducible curves whose components are non-singular and rational, the condition for psd=sos will depend on the configuration of the irreducible components, so we need some combinatorial preparations: We associate with a curve C (over any field k) a finite graph Γ_C as follows: The vertices of Γ_C are the irreducible components of C, and we put an edge between two distinct vertices for every intersection point of the corresponding components. (The definition of Γ_C has been changed compared to [8] after a suggestion by Michel Coste, simplifying the arguments that follow.) Recall that a *simple cycle* of a graph is a subgraph that is homeomorphic to S^1. A graph that does not contain any simple cycles is called a *forest* (a *tree* if it is also connected).

Lemma 3.10. *Let C be a curve k with irreducible components $C_1, \dots, C_m$. Then the graph Γ_C is a forest if and only if $C_1, \dots, C_m$ can be relabelled in such a way that $C_i \cap (C_1 \cup \cdots \cup C_{i-1})$ consists of at most one point for every $1 < i \leqslant m$.*

Proof. Clearly, we may assume that C is connected, otherwise we can treat all connected components separately. We prove the result by induction on m. The case $m = 1$ is trivial, so assume that $m \geqslant 2$. It is elementary that a finite connected graph with m vertices is a tree if and only if it has exactly $m-1$ edges. It follows that if Γ_C is a tree, there exists a vertex of degree one. We may relabel and assume that this vertex corresponds to the component C_m. Put $C' = C_1 \cup \cdots \cup C_{m-1}$. Then $C_m \cap C'$ consists of exactly one point. Furthermore, $\Gamma_{C'}$ is again a tree, so we are done by the induction hypothesis. Conversely, assume that $C_1, \dots, C_m$ are arranged such that $C_i \cap (C_1 \cup \cdots \cup C_{i-1})$ consists of exactly one point for every $1 < i \leqslant m$. Again, write $C' = C_1 \cup \cdots \cup C_{m-1}$. Then $\Gamma_{C'}$ is a tree by the induction hypothesis, hence it has $m-1$ vertices and $m-2$ edges. Since $C_m \cap C'$ consists of exactly one point, it follows that Γ_C has m vertices and $m-1$ edges, so it is a tree. □

If C is a real curve with only real intersection points, the condition on Γ_C can sometimes be expressed in terms of the semialgebraic topology of $C(R)$:

Proposition 3.11. *Let R be a real closed field, and let C be a curve over R with irreducible components $C_1, \dots, C_m$. Assume that all intersection points of C are real and that $C_i(R)$ is simply connected for all $1 \leqslant i \leqslant m$. Then Γ_C is a forest if and only if every connected component of $C(R)$ is simply-connected.*

Proof. It suffices to note that, under the hypotheses, cycles in Γ_C correspond exactly to non-trivial 1-cycles of $C(R)$. □

Example 3.12. Even for real curves, it does not suffice to take only the real picture into account if the $C_i(R)$ are not connected. For example, let C be the plane curve $\{(xy-1)(x-y) = 0\}$, a hyperbola intersecting a line. Clearly, $C(R)$ is simply connected, yet Γ_C is a simple cycle consisting of two vertices joint by two edges.

Theorem 3.13. *Let C be an affine curve over R with irreducible components $C_1, \dots, C_m$, and assume that $B(C_i) = R$ for all $i = 1, \dots, m$. Then psd=sos in $R[C]$ if and only if the following conditions are satisfied:*

1. *All C_i are isomorphic to open subcurves of $\mathbb{A}^1_R$.*
2. *All intersection points of C are real ordinary multiple points with independent tangents.*
3. *The graph Γ_C is a forest.*

Proof. Assume that C satisfies all the conditions listed in the theorem. Condition (1) implies that psd=sos in $R[C_1], \dots, R[C_m]$ by Thm. 3.9. Condition (3) implies by

Lemma 3.10 that we may rearrange the C_i in such a way that $C_i \cap (C_1 \cup \cdots \cup C_{i-1})$ consists of at most one point for all $1 \leqslant i \leqslant m$. Put $E_i = C_1 \cup \cdots \cup C_i$ for all $1 \leqslant i \leqslant m$ and use induction on i: We already know that psd=sos in $R[E_1]$. For $i \geqslant 2$, psd=sos in $R[C_i]$ and in $R[E_{i-1}]$ by the induction hypothesis. Now C_i and E_{i-1} have at most one intersection point which must then be a real ordinary multiple point with independent tangents, by condition (2). Therefore,

$$R[E_i] = \begin{cases} R[C_i] \times R[E_{i-1}], & C_i \cap E_i = \emptyset \\ \bigl\{(f,g) \in R[C_i] \times R[E_{i-1}] \bigm| f(P) = g(P)\bigr\}, & C_i \cap E_i = \{P\} \end{cases}$$

by Cor. 1.7. Thus psd=sos in $R[E_i]$ by Prop. 2.2. Eventually, we reach $i = m$, and we see that psd=sos in $R[E_m] = R[C]$.

For the converse, assume that psd=sos in $R[C]$. By Prop. 3.3, psd=sos in $R[C_1], \ldots, R[C_m]$, so Thm. 3.9 implies condition (1). Furthermore, Prop. 3.1 and Prop. 3.4 imply condition (2). Thus we are left with the case when conditions (1) and (2) are satisfied, but (3) is not, i.e. the graph Γ_C contains a simple cycle. Let C_i be an irreducible component of C corresponding to a vertex in a simple cycle of Γ_C. Let $C_i' = \bigcup_{j \neq i} C_j$ as before. Then there is a connected component E of C_i' such that $E \cap C_i$ contains at least two distinct points. It suffices to show that psd$\neq$sos in $R[C_i \cup E]$ by Lemma 3.3. So we may replace C by $C_i \cup E$ and assume right away that C and C_i' are connected. Write $C_i \cap C_i' = \{P_1, \ldots, P_r\}$, $r \geqslant 2$, and let

$$A = \bigl\{f \in R[C_i] \bigm| f(P_1) = \cdots = f(P_r)\bigr\}.$$

From Cor. 1.7 and the fact that C_i' is connected, we see that the restriction map $R[C] \to R[C_i]$ induces an isomorphism

$$\bigl\{f \in R[C] \bigm| f|_{C_i'} \text{ is constant}\bigr\} \xrightarrow{\sim} A.$$

Now if an element of A is a sum of squares in $R[C]$, then it is a sum of squares in A by Lemma 1.12 (7). (Note that since C_i' is connected, $B(C_i) = R$ for all $i = 1, \ldots, r$ implies $B(C_i') = R$, by Lemma 1.12 (5).)

We will make a similar argument as in the proof of Prop. 3.4 and construct an element of A that is not a sum of squares in A as follows: Fix an embedding $C_i \hookrightarrow \mathbb{P}^1_R$, and let $\mathbb{P}^1_R \setminus C_i = \{Q_1, \ldots, Q_s\}$ be the points at infinity of C_i. Since $B(C_i) = R$, all Q_j are real by Lemma 1.12 (3). Furthermore, since $\mathbb{P}^1(R)$ is topologically a circle, we can relabel $P_1, \ldots, P_r$ and assume that P_j is next to P_{j+1}, i.e. if U_1 and U_2 are the two connected components of $\mathbb{P}^1(R) \setminus \{P_j, P_{j+1}\}$, either U_1 or U_2 contains none of $P_1, \ldots, P_r$, for $1 \leqslant j \leqslant r-1$. We denote that connected component by (P_j, P_{j+1}). Assume further that $Q_1 \in (P_r, P_1)$.

Fix $j \in \{1, \ldots, r\}$. Since C_i is rational, there exists $h_j \in R[C_i]$ such that

$$\operatorname{div}_{\mathbb{P}^1_R}(h_j) = P_1 + \cdots + \widehat{P}_j + \cdots + P_r - (r-1)Q_1.$$

Then $h_j(P_j) \neq 0$, and after multiplying with $(-1)^j \cdot h(P_j)^{-1}$ we can assume that $h_j(P_j) = (-1)^j$. Put $f = \sum_j h_j$; then $f(P_j) = (-1)^j$. We have $\operatorname{ord}_{Q_1}(f) \geqslant -(r-1)$ and $\operatorname{ord}_P(f) \geqslant 0$ for all $P \in \mathbb{P}^1_R$, $P \neq Q_1$, so f has poles only at Q_1, and f can have at most $r-1$ distinct zeros. But f changes sign on (P_j, P_{j+1}), so it must have a zero $R_j \in (P_j, P_{j+1})$ for every $j = 1, \ldots, r-1$. We conclude that

$$\operatorname{div}_{\mathbb{P}^1_R}(f) = R_1 + \cdots + R_{r-1} - (r-1)Q_1.$$

Now since $f^2(P_j) = 1$ for all $1 \leqslant j \leqslant r$, we have $f^2 \in A$, but $f \notin A$, since $f(P_j) = (-1)^j$ for $j = 1, \ldots, r$ and $r \geqslant 2$. We will show that f^2 cannot be a sum of squares in A: For if $f^2 = \sum f_j^2$ with $f_j \in R[C_i]$, then $f_j(R_l) = 0$ for every

$1 \leqslant l \leqslant r-1$ and every j, so each f_j has at least $r-1$ distinct zeros. On the other hand, $\mathrm{ord}_{Q_l}(f_j) \geqslant \mathrm{ord}_{Q_l}(f)$ for $l = 1, \dots, s$ by Lemma 1.9, so $\mathrm{ord}_{Q_1}(f_j) \geqslant -(r-1)$ and $\mathrm{ord}_P(f_j) \geqslant 0$ for all $P \in \mathbb{P}^1_R$, $P \neq Q_1$. It follows that $\mathrm{div}_{\mathbb{P}^1_R}(f_j) = R_1 + \cdots + R_{r-1} - (r-1)Q_1 = \mathrm{div}_{\mathbb{P}^1_R}(f)$, so there exists $c_j \in R$ such that $f_j = c_j f$, hence $f_j \notin A$. □

Corollary 3.14. *Let C be an affine curve over R all of whose irreducible components are isomorphic to $\mathbb{A}^1_R$. Then psd=sos in $R[C]$ if any only if the following conditions are satisfied:*

(1) *All intersection points of C are real ordinary multiple points with independent tangents.*
(2) *All connected components of $C(R)$ are simply connected.*

Proof. Combine the theorem with Prop. 3.11. □

Combining Thm. 3.13 with the preceding results, we can now treat the general case of a reducible curve C over $\mathbb{R}$ with irreducible components $C_1, \dots, C_m$ where $B(C_i) = \mathbb{R}$ holds for some components of C while $B(C_i) \neq \mathbb{R}$ holds for others:

Theorem 3.15. *Let C be an affine curve over $\mathbb{R}$, and let C' be the union of all irreducible components C_i of C such that $B(C_i) = \mathbb{R}$. Then psd=sos in $\mathbb{R}[C]$ if and only if the following conditions are satisfied:*

(1) *All real singularities of C are ordinary multiple points with independent tangents.*
(2) *All intersection points of C are real.*
(3) *All irreducible components of C' are isomorphic to open subcurves of $\mathbb{A}^1_{\mathbb{R}}$.*
(4) *The graph $\Gamma_{C'}$ is a forest.*

Proof. Let C_r be the union of all irreducible components of C that are real, C_{nr} the union of those that are non-real. Condition (1) implies that $C_{nr}(\mathbb{R}) = \emptyset$ and condition (2) implies that $C_r \cap C_{nr} = \emptyset$, so if (1)–(4) are satisfied, psd=sos in $\mathbb{R}[C]$ is equivalent to psd=sos in $\mathbb{R}[C_r]$ by Cor. 2.10. Conversely, if psd=sos in $\mathbb{R}[C]$, then $C_{nr}(\mathbb{R}) = \emptyset$ and $C_r \cap C_{nr} = \emptyset$ hold by 2.10, so that (1)–(4) are satisfied for C if and only if they are satisfied for C_r. We may therefore assume that $C = C_r$, i.e. C is real.

The necessity of (1) is Prop. 3.1, that of (2) is 3.4, that of (3) and (4) follows from Thm. 3.13. Conversely, assume that (1)–(4) are satisfied: Let C'' be the union of all irreducible components C_i of C such that $B(C_i) \neq \mathbb{R}$. Then $C''(\mathbb{R})$ is virtually compact, hence psd=sos in $\mathbb{R}[C'']$ by Cor. 3.7. Also, psd=sos in $\mathbb{R}[C']$ by Thm. 3.13. Thus psd=sos in $\mathbb{R}[C]$ by Thm. 2.7. □

4. Preorderings on curves

The case of general preorderings instead of just sums of squares is substantially harder, already in the irreducible case. (See 1.15 for basic notations and definitions used in this section). As far as the irreducible case is concerned, we content ourselves here with citing the results of Scheiderer for non-singular curves and the results of Kuhlmann and Marshall for subsets of the line:

Theorem 4.1 (Scheiderer [12], Thm. 5.17 and [11], Thm. 3.5)**.** *Let C be a non-singular, irreducible affine curve over $\mathbb{R}$, $\mathcal{H}$ a finite subset of $\mathbb{R}[C]$, $T = \mathrm{PO}(\mathcal{H})$, and $S = \mathcal{S}(T)$. Assume that $B_C(S) \neq \mathbb{R}$ holds. Then T is saturated if and only if the following conditions are satisfied:*

(1) *For every boundary point P of S in $C(\mathbb{R})$ there exists an element $h \in \mathcal{H}$ such that* $\operatorname{ord}_P(h) = 1$.
(2) *For every isolated point P of S there exist $h_1, h_2 \in \mathcal{H}$ such that* $\operatorname{ord}_P(h_1) = \operatorname{ord}_P(h_2) = 1$ *and $h_1 h_2 \leqslant 0$ holds in a neighbourhood of P in $C(\mathbb{R})$.*

The results in the singular case are more complicated to state, and we refer the reader to Scheiderer [12]. The complementary case $B_C(S) = \mathbb{R}$ is covered by the following result:

Theorem 4.2 (Scheiderer [11], Thm. 3.5)**.** *Let C be a non-singular, irreducible affine curve over $\mathbb{R}$, and let S be a basic closed subset of $C(\mathbb{R})$ such that $B_C(S) = \mathbb{R}$. Then the preordering $\mathcal{P}_C(S)$ is finitely generated if and only if C is an open subcurve of $\mathbb{A}^1_{\mathbb{R}}$.*

In the case of the affine line, it is possible to say precisely what the generators of the saturated preordering $\mathcal{P}_C(S)$ must look like:

Theorem 4.3 (Kuhlmann-Marshall [4], Thm 2.2)**.** *Let $\mathcal{H}$ be a finite subset of $\mathbb{R}[t]$, $T = \mathrm{PO}(\mathcal{H})$, and $S = \mathcal{S}(T)$. Assume that $B(S) = \mathbb{R}$ (i.e. S is not compact). Then T is saturated if and only if the following hold:*
(1) *If $a = \min(S)$ exists, then $\lambda(t-a) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbb{R}$.*
(2) *If $a = \max(S)$ exists, then $\lambda(a-t) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbb{R}$.*
(3) *If $a < b \in S$ are such that $(a,b) \cap S = \emptyset$, then $\lambda(t-a)(t-b) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbb{R}$.*

Properties (1)–(3) of the theorem specify a minimal set of generators for T (unique up to positive scalars) that Kuhlmann and Marshall call the *natural generators.*

We now turn our attention to reducible curves, but will only treat the simplest case and some examples. We need the following

Lemma 4.4. *Let C be a connected affine curve over R with irreducible components $C_1, \ldots, C_m$. Assume that all intersection points of C are ordinary multiple points with independent tangents and that the graph Γ_C is a forest. Then for all $i = 1, \ldots, m$ and all $f \in R[C]$, there exists a unique function $f_{(i)} \in R[C]$ such that $f_{(i)}|_{C_i} = f|_{C_i}$ and $f_{(i)}|_{C_j}$ is constant for all $j \neq i$.*

Proof. The claim is trivial for $m = 1$, so assume $m \geqslant 2$ and put $C' = C_1 \cup \cdots \cup C_{m-1}$. By Lemma 3.10, we may relabel and assume that $C_m \cap C'$ consists of a single point P. Now $R[C] = \big\{(f, f') \in R[C_m] \times R[C'] \,\big|\, f(P) = f'(P)\big\}$ by Cor. 1.7, so for every $(f, f') \in R[C]$ we can form $(f'(P), f') \in R[C]$ and $(f, f(P)) \in R[C]$. From this the claim follows easily by induction. □

Proposition 4.5. *Let C be a connected affine curve over R with irreducible components $C_1, \ldots, C_m$, let $\mathcal{H} \subset R[C]$ be a finite subset, and let $S = \mathcal{S}(\mathcal{H})$ and $T = \mathrm{PO}(\mathcal{H})$. Assume that the following conditions are satisfied:*
(1) *The induced preordering $T|_{C_i}$ is saturated for all $i = 1, \ldots, m$.*
(2) *All intersection points of C are real ordinary multiple points with independent tangents and are contained in S.*
(3) *The graph Γ_C is a tree.*
(4) *If $h \in \mathcal{H}$, then $h_{(i)} \in \mathcal{H}$ for all $i = 1, \ldots, m$, where $h_{(i)}$ is defined as in the preceding lemma.*

Then T is saturated.

Proof. The claim is trivial for $m=1$ because of condition (1), so assume $m \geqslant 2$. Write $C = C_m \cup C'$, $C_m \cap C' = \{P\}$, as in the proof of the preceding lemma using hypotheses (2) and (3), and write $\mathcal{H} = \{H_1, \ldots, H_r\}$, $H_i = (h_i, h_i')$. We first show that for every $f \in R[C_m]$ with $f \geqslant 0$ on $S \cap C_m(R)$, the function $F_{(m)} = (f, f(P)) \in R[C]$ is contained in T. By hypothesis (1), f has a representation $f = \sum_i s_i \underline{h}^i$ with $s_i \in \sum R[C_m]^2$. Thus we can write $(f, f(P)) = \sum_i (s_i, s_i(P))(\underline{h}^i, \underline{h}^i(P)) = \sum_i (s_i, s_i(P)) \underline{H}^i_{(m)}$, and $\underline{H}^i_{(m)} \in \mathcal{H}$ by hypothesis (4); thus $(f, f(P)) \in T$, as claimed. In the same way, we show $(f'(P), f') \in T$ for every $f' \in R[C']$ with $f' \geqslant 0$ on $S \cap C'(R)$, using the induction hypothesis instead of (1).

Now let $F = (f, f') \in R[C]$, and assume that $F \geqslant 0$ holds on S. If $F(P) = 0$, then $F = (f, 0) + (0, f') \in T$ by what we have just shown. If $F(P) \neq 0$, then $F(P) > 0$ by hypothesis (2), and we can write $F = (f, f(P))(1, f(P)^{-1} f') \in T$. □

Remark. Note that we also could have proved Prop. 2.2 by the same method employed here.

Examples 4.6. (1) Let C be the plane curve $\{xy = 0\}$ in $\mathbb{A}^2_R$. Write $R[C] = \{(f, g) \in R[u] \times R[v] \mid f(0) = g(0)\}$, and consider the preordering $T = \mathrm{PO}((u, 0), (0, v))$ defining the semialgebraic set $S = \mathcal{S}(T) = \{(x, y) \in C(R) \mid x \geqslant 0 \wedge y \geqslant 0\}$. By the proposition, T is saturated. However, the proposition does not apply to the preodering $\mathrm{PO}((u, v))$ that defines the same set, because condition (4) is violated. In fact, $(u, 0) \notin \mathrm{PO}((u, v))$ since $(u, 0) = (s_1, t_1) + (s_2, t_2)(u, v)$ with $s_i \in \sum R[u]^2$ and $t_i \in \sum R[v]^2$ would imply $t_1 = t_2 = s_1 = 0$ by degree considerations and $s_2 = 1$, but $(1, 0) \notin R[C]$. Thus $\mathrm{PO}((u, v))$ is not saturated.

(2) Let C be as before, and let $T = \mathrm{PO}((u^2 - 1, v^2 - 1))$, so that $S = \mathcal{S}(T) = \{(x, y) \in C(R) \mid |x| \geqslant 1 \wedge |y| \geqslant 1\}$. Here, the intersection point $(0, 0)$ is not contained in S, and the proposition does not apply. In fact, T is not saturated: Let $f \in R[u]$ be any quadratic polynomial that is non-negative on $S \cup \{(0, 0)\}$ but not psd (i.e. not a sum of squares in $R[u]$). For example, $f = (u - \frac{1}{4})(u - \frac{3}{4})$ will do. Then $(f, f(0))$ is a function in $R[C]$ that is non-negative on S but not contained in T: For $(f, f(0)) = (s_1, t_1) + (s_2, t_2)(u^2 - 1, v^2 - 1)$ would imply $t_1 = f(0)$ and $t_2 = 0$ by degree considerations, so we must have $s_2(0) = 0$. On the other hand, $s_2 \in R$ again because of the degree of f, so $(f, f(0)) = (s_1, t_1)$, a contradition. It is not clear whether the preordering $\mathcal{P}_C(S)$ is finitely generated. The fact that the intersection point is not contained in S makes condition (4) of the proposition unfulfillable, and it is not clear what a suitable replacement should look like. The best guess for a set of generators in this particular instance seems to be $\{(u^2 - 1, v^2 - 1), (u(u-1), 0), (u(u+1), 0), (0, v(v-1)), (0, v(v+1))\}$, but I have been unable to verify this.

5. Applications to the moment problem

Given a closed subset $K \subset \mathbb{R}^n$, the existence part of the K-moment problem of functional analysis asks for a characterisation of those linear functionals $L \colon \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}$ for which there exists a (positive) Borel-measure μ supported on K such that $L(f) = \int_K f d\mu$ holds for all $f \in \mathbb{R}[x_1, \ldots, x_n]$. Let V be a real affine algebraic variety, and let $K = \mathcal{S}(h_1, \ldots, h_r)$ be a basic closed semialgebraic subset of $V(\mathbb{R})$, defined by elements $h_1, \ldots, h_r \in \mathbb{R}[V]$; let $T = \mathrm{PO}(h_1, \ldots, h_r)$ be the corresponding finitely generated preordering of $\mathbb{R}[V]$. In an algebraic reformulation of the K-moment problem (see for example Schmüdgen [17], and Powers-Scheiderer

[9]), one says that T has the *strong moment property* (SMP) if $L|_T \geqslant 0$ implies $L|_{\mathcal{P}(K)} \geqslant 0$ for every linear functional $L\colon \mathbb{R}[V] \to \mathbb{R}$. In other words, T has the strong moment property if the cone T and the psd cone of K in $\mathbb{R}[V]$ cannot be separated by any linear functional. In particular, any saturated preordering has property (SMP). A classical result of Haviland says that $L|_{\mathcal{P}(S)} \geqslant 0$ is necessary and sufficient for the existence of a measure representing L. So if T has the strong moment property, this condition is replaced by the more manageable condition $L|_T \geqslant 0$. Apart from the functional-analytic importance, the strong moment property can be seen simply as an approximation property for psd elements by elements of T. It is a consequence of Schmüdgen's Positivstellensatz that T has the strong moment property whenever K is compact; see [16]. In 2004, Schmüdgen went on to prove the following:

Theorem 5.1 (Schmüdgen's fibration theorem [17])**.** *Let V be an affine $\mathbb{R}$-variety, T a finitely generated preordering of $\mathbb{R}[V]$, and $K = \mathcal{S}(T)$. Assume that $\varphi\colon V \to \mathbb{A}^m$ is a real polynomial map such that the closure of $\varphi(K)$ in $\mathbb{R}^m$ is compact. Then T has property (SMP) if and only if $T|_{\varphi^{-1}(a)}$ has property (SMP) for all $a \in \mathbb{R}^m$.*

Schmüdgen's proof relies heavily on operator theory. In the meantime, Marshall and Netzer have (independently) developed more elementary proofs; see Netzer [7] and Marshall [6], Ch. 4.

Note that $T|_{\varphi^{-1}(a)}$ denotes the restriction of T to the reduced fibre $\varphi^{-1}(a)$, i.e. if we write $a = (a_1, \dots, a_m)$, $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i \in B_V(K)$, and $I_a = (\varphi_i - a_i \mid i \in \{1, \dots, m\})$, then $\varphi^{-1}(a) = \mathcal{V}(\sqrt{I_a})$ so that $T|_{\varphi^{-1}(a)} = (T + \sqrt{I_a})/\sqrt{I_a}$. In other words, $T|_{\varphi^{-1}(a)}$ is the preordering induced by T in the coordinate ring $\mathbb{R}[\varphi^{-1}(a)] = \mathbb{R}[V]/\sqrt{I_a}$ of the fibre. Schmüdgen states his theorem for $T + I_a$, and some additional effort is required to pass from his version to the one above; see Scheiderer [13], section 4, or [8], Lemma 2.3.

We can apply Schmüdgen's fibration theorem to reduce the moment problem for curves to the case where there do not exist any non-constant bounded functions. In that case, the moment property is equivalent to saturatedness, by a result of Powers and Scheiderer.

Proposition 5.2. *Let C be an affine curve over $\mathbb{R}$, let T be a finitely generated preordering of $\mathbb{R}[C]$, and put $K = \mathcal{S}(T)$. Let C' be the union of all irreducible components C_i of C for which $B_{C_i}(K \cap C_i(\mathbb{R})) = \mathbb{R}$. The following are equivalent:*

(1) *T has the strong moment property;*
(2) *$T|_{C'}$ has the strong moment property;*
(3) *$T|_{C'}$ is saturated.*

Proof. (1) implies (2) since the moment property is preserved when passing to a closed subvariety; see [13], Prop. 4.6. By Thm. 2.14 in Powers and Scheiderer [9], $T|_{C'}$ is closed in $\mathbb{R}[C']$ (with respect to the finest locally convex topology). Therefore, saturatedness and the strong moment property are equivalent for $T|_{C'}$. Finally, (2) implies (1). For if C_i is any component of C such that $B_{C_i}(K \cap C_i(\mathbb{R})) \neq \mathbb{R}$, put $C'' = C' \cup C_i$. By Lemma 1.12 (5), we may choose $f \in \mathbb{R}[C'']$ with $f|_{C_i} \in B_{C_i}(K \cap C_i(\mathbb{R})) \setminus \mathbb{R}$ and $f|_{C'} = 0$. The fibres of $f\colon C'' \to \mathbb{A}^1_{\mathbb{R}}$ are the points of C_i and the curve C', so Schmüdgen's fibre theorem implies that $T|_{C''}$ has the strong moment property. Now continue inductively until $C'' = C$. $\square$

Combined with Thm. 3.13, this proposition gives a complete set of necessary and sufficient conditions for the preordering $\sum \mathbb{R}[C]^2$ of sums of squares on an

affine curve C over $\mathbb{R}$ to have the strong moment property. Sufficient conditions for general preorderings on curves can be derived from Prop. 4.5.

If V is an irreducible affine surface and $K \subset V(\mathbb{R})$ is such that there exists a non-constant bounded polynomial map $\varphi\colon K \to \mathbb{R}^m$, then the fibres of φ are curves and Schmüdgen's fibre theorem can be nicely combined with results for curves to say something about the moment problem for K. We want to stress here that reducible curves come up most naturally in this context: Even if V and φ have good properties, one cannot expect all fibres of φ to be irreducible. Two examples:

Examples 5.3. (1) Let T be the preordering of $\mathbb{R}[x,y]$ generated by $1-x^2y^2$, and put $K = \mathcal{S}(T)$. The map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\varphi(x,y) = xy$ is obviously bounded on K. For $a \in \mathbb{R}$, put $C_a = \varphi^{-1}(a)$. All fibres $C_a = \mathcal{V}(xy-a)$ are hyperbolas for $0 \neq a \in [-1,1]$ and $T|_{C_a} = \sum \mathbb{R}[C_a]^2$. Since psd=sos in $\mathbb{R}[C_a]$ (Thm. 3.9), all T_{C_a}, $0 \neq a \in [-1,1]$, have property (SMP). The reducible fibre $C_0 = \mathcal{V}(xy)$ is a pair of crossing lines and the induced preordering $T|_{C_0}$ is again just $\sum \mathbb{R}[C_0]^2$ so that $T|_{C_0}$ has property (SMP), too. Hence so does T by Schmüdgen's fibration theorem.

(2) If we restrict to the first quadrant in the previous example, i.e. if we consider $T = \mathrm{PO}(x, y, 1-xy)$, the same argument shows that T has property (SMP). Here, we have to use the fact that $T|_{C_0}$ is the preordering generated by x and y: Under the natural isomorphism $\mathbb{R}[C_0] = \mathbb{R}[x,y]/(xy) \cong \mathbb{R}[t] \times \mathbb{R}[u]$, this means that $T|_{C_0}$ is the preordering generated by $(t,0)$ and $(0,u)$ which is saturated (see Example 4.6 (1)).

On the other hand, if $T = \mathrm{PO}(x+y, 1-x^2y^2)$, then T does not have property (SMP), since the restriction $T|_{C_0}$ is the preordering generated by (t,u) which is not saturated by the discussion in Example 4.6 (1). It follows that $T|_{C_0}$ cannot have property (SMP) either.

References

[1] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, vol. 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1998.

[2] N. Bourbaki. *Éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.* Actualités Scientifiques et Industrielles, No. 1293. Hermann, Paris, 1961.

[3] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977.

[4] S. Kuhlmann and M. Marshall. Positivity, sums of squares and the multi-dimensional moment problem. *Trans. Amer. Math. Soc.*, **354 (11)**, 4285–4301, 2002.

[5] S. Kuhlmann, M. Marshall, and N. Schwartz. Positivity, sums of squares and the multi-dimensional moment problem. II. *Adv. Geom.*, **5 (4)**, 583–606, 2005.

[6] M. Marshall. *Positive polynomials and sums of squares*, vol. 146 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2008.

[7] T. Netzer. An elementary proof of Schmüdgen's theorem on the moment problem of closed semi-algebraic sets. *Proc. Amer. Math. Soc.*, **136 (2)**, 529–537 (electronic), 2008.

[8] D. Plaumann. Bounded polynomials, sums of squares, and the moment problem. PhD thesis, Universität Konstanz, 2008.

[9] V. Powers and C. Scheiderer. The moment problem for non-compact semialgebraic sets. *Adv. Geom.*, **1 (1)**, 71–88, 2001.

[10] A. Prestel and C. N. Delzell. *Positive polynomials.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001. From Hilbert's 17th problem to real algebra.

[11] C. Scheiderer. Sums of squares of regular functions on real algebraic varieties. *Trans. Amer. Math. Soc.*, **352 (3)**, 1039–1069, 2000.

[12] ———. Sums of squares on real algebraic curves. *Math. Z.*, **245 (4)**, 725–760, 2003.

[13] ———. Non-existence of degree bounds for weighted sums of squares representations. *J. Complexity*, **21 (6)**, 823–844, 2005.

[14] ———. Sums of squares on real algebraic surfaces. *Manuscripta Math.*, **119 (4)**, 395–410, 2006.

[15] ———. Positivity and sums of squares: A guide to recent results. In M. Putinar and S. Sullivant (editors), *Emerging Applications of Algebraic Geometry*, vol. 149 of *IMA Volumes Math. Appl.*, pp. 271–324. Springer, 2009.

[16] K. Schmüdgen. The K-moment problem for compact semi-algebraic sets. *Math. Ann.*, **289 (2)**, 203–206, 1991.

[17] ———. On the moment problem of closed semi-algebraic sets. *J. Reine Angew. Math.*, **558**, 225–234, 2003.

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